

Optimal Passive Dynamics for Physical Interaction: Throwing a Mass

Hamid Reza Vejdani and Jonathan W Hurst

*Dynamic Robotics Laboratory
Oregon State University
Corvallis, Oregon*

Abstract—The passive dynamics of actuators may impose serious limitations to the performance of a system. Existence of inertia for example makes it impossible for the actuators to react immediately. A throwing mechanism (with electric motors) is composed of two inertias (object and motor) that decreases the performance of the system and can not be overcome with software control. But, we can use other elements (like a spring) to make the motor inertia a benefit to improve the performance of the system. Moreover, when the object is directly connected to the motor, the maximum velocity that the object can achieve is limited to the maximum velocity that can be provided by the motor. Previous research shows that passive dynamics is not always harmful, and can increase the performance of a mechanism. Here, we will extract mathematical formula that gives us the required optimum value for stiffness and/or damping of the system to give us the optimal performance given physical limitations.

I. INTRODUCTION

Physical interaction tasks like catching and throwing (i.e. jumping and landing) are done by animals much better than robots. Although rigid robots are very good at some tasks like accurate positioning of objects, they perform poorly in accomplishing physical interaction tasks. We think that most of the amazing performance of animals is due to the physical characteristics of their mechanical systems and their synchronicity of their control policy.

Throwing an object directly by a rigid robot which is driven by electric motor is limited to the maximum velocity that the motor can provide. In this case, to reach the maximum possible velocity, the robot applies the maximum force (or torque) to the system until either the motor reaches the limit of its range of motion or the object reaches the maximum velocity of the motor. By using the characteristics of springs and dampers, the performance of the system can be improved significantly.

The idea of using spring for throwing a mass is to store energy in the spring in the beginning of the process to help the motor push the object faster while the motor is at its maximum velocity. In this process, the inertia of the motor helps the system to accelerate the object even more. Since the idea is to transfer as much energy as possible to the object, the existence of damper which dissipates energy seems destructive. However, considering the other mechanical limitations like maximum motor range of motion

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The authors can be contacted by email at:

hrvn@enr.oregonstate.edu

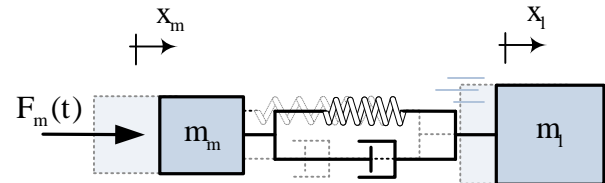


Fig. 1. The system we investigate in this paper is entirely linear and includes damping, elasticity, motor inertia (represented as an equivalent mass), motor force limits and motor maximum velocity as well as maximum spring compression length.

or maximum allowable spring compression, the existence of damping can become beneficial.

Regardless of the software and controller, there are some physical limitations that impose serious limits. The motor inertia which is amplified through the gearbox or the maximum distance that the motor can travel are among the physical limitations that the software can not overcome. In the other words, no matter which controller is used, the motor can not travel further than its maximum limit or the motor can not respond instantly and generate the desired velocity. In this paper we use an schema for throwing an object and present formulations to calculate the optimal values for the parameters of the new system (elasticity and damping) to have an optimal physical performance. In the mathematical framework of our mechanical system we consider inertia, torque limit and velocity limit for the electric motor in series with a spring-damper system (which has also compression limit) as shown in figure 1.

II. BACKGROUND

The motivation of this paper is to investigate the effect of elasticity and damping on initializing the process of running for a legged robot from its rest position using electric motors. Other researchers have used the subject of throwing for robot's arms [1][2], hopping [3] or as a new method for transportation of objects[4][5].

The mechanism of running and walking in animals can be best presented by spring-mass model [6][7]. Although roboticists who have built machines to mimic spring-like behavior [8][9][10] acknowledged that elasticity provides robustness, but their studies focused on energy storage and efficiency. Little attention is given to how these elements contribute to general force control and manipulation with the environment. Recently [3] investigated the effect of

compliant actuator on the energy efficiency of a hopping robot. They concluded that series elastic elements help the robot to achieve higher hopping height.

Early investigations into force control found that series compliance in an actuator can increase stability, and in some cases is required for stable operation [11]. Researchers at the Massachusetts Institute of Technology (MIT) Leg Laboratory explored these ideas and created the Series Elastic Actuator (SEA). The MIT-SEA is designed specifically to include an elastic element as a force sensor and low impedance coupling between the drive system and the load to improve force control. It has been shown that this configuration provides filtering to handle shock loads and higher bandwidth force control [12].

Hurst et al. [13] proposed an extension to MIT-SEA. They investigated the effect of damping and concluded that the added damping provides higher bandwidth than a purely series-elastic element. But, initial observed force by the drive system at impact is greater than a system that is only composed of an elastic element.

Haddadin et al. [14] showed that it is possible to derive suitable stiffness for an elastic joint and it is capable of at least reaching the maximum velocity of the rigid joint.

Braun et al. [15] proposed an optimal stiffness profiles for a variable stiffness system. They chose throwing a ball to demonstrate their method. Garabini et al. [16] also investigated the optimality principles in stiffness control. They imposed a fixed terminal time in their optimization program to maximize the velocity of the actuator link.

Throwing an object has been considered a means in transporting objects[5] [4]. Frank et. al. in [5], used a rigid rotary system to throw the objects. A rotary electric motor is connected to the rigid arm with a specific length (which is determined based on the desired final velocity) and the mass is located at the end of the arm. To increase the final velocity of the thrown objects, the length of the arm should be increased which increases the inertia of the system quadratically ($I = I_0 + m \cdot d^2$).

In this paper we show quantitatively how the spring and damping affect the behavior of the throwing mechanism. Moreover, we will present mathematical formulations to relate the various parameters of the actuator. It will be shown that for various physical limitations (such as motor/spring stroke) there is an optimum value for stiffness and damping that gives us the greatest final velocity.

III. PROBLEM DEFINITION

To investigate the effect of elasticity and damping on the performance of throwing, the system in Fig. 2 is considered. In the mathematical model of the system, in addition to the elements k and B , we include motor force limits, motor inertia and motor velocity limit as well as maximum spring compression length. The motor torque and rotational inertia are modeled as a linear mass with applied force (similar to a ballscrew). The following symbols describe our mathematical model:

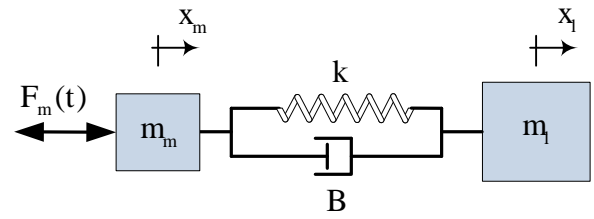


Fig. 2. System schematic. The motor inertia is represented as a linear mass (m_m) and the load mass is represented as (m_l). This is analogous to an electric motor attached to a ballscrew transmission where the rotational inertia is much greater than the mass of the transmission itself.

k	Spring constant	$N \cdot m$
B	Damping constant	$\frac{N \cdot s}{m}$
m_m	Motor/transmission mass	kg
m_L	Load mass	kg
F_m	Motor force	N
F_{limit}	Motor force limit	N
F_d	Force caused by the dynamic elements	N
v_{max}	Motor maximum velocity	$\frac{m}{s}$

We assume that the actuator can not sustain tension (like jumping) therefore the final velocity of the object is the velocity that it has at the first loss of contact (unlike the case for pogo sticks that to reach the maximum velocity, it losses the contact several times). Our goal in this paper is to show how to calculate the optimum values for stiffness and damping to be added to the passive dynamics of the system to improve the performance the most (means maximizes the final velocity of the object).

IV. MATHEMATICAL FORMULATION

To model the effects of the passive dynamics of an actuator to the performance of our system, the system shown in Fig. 2 is considered. We want to know how to choose the values of elasticity and damping (k and B) to have the most efficient throwing system given our physical limitations like stroke limit. Moreover, the system shown in Fig. 2 is similar to the mechanism of legged robots [8] [9].

We define the performance of the system as the largest possible v that the object can reach without breaking the contact to the system given the physical limitations. In our model, the spring is linear and the damper is a viscous damper, therefore the dynamical behavior of the system is linear. The differential equations that describe the motion of the system are:

$$[m] \begin{Bmatrix} \ddot{x}_l \\ \ddot{x}_m \end{Bmatrix} + [B] \begin{Bmatrix} \dot{x}_l \\ \dot{x}_m \end{Bmatrix} + [k] \begin{Bmatrix} x_l \\ x_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_m(t) \end{Bmatrix} \quad (1)$$

where

$$[B] = \begin{bmatrix} B & -B \\ -B & B \end{bmatrix} \quad (2)$$

$$[k] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \quad (3)$$

$$[m] = \begin{bmatrix} m_l & 0 \\ 0 & m_m \end{bmatrix} \quad (4)$$

Here, the $[B]$, $[k]$ and $[m]$ are respectively damping, stiffness and mass matrices. As the system of differential equations (eq. 1) is coupled, it can not be solved in this form. To solve the system, we decoupled (1) into two independent single degree of freedom (SDOF) systems using the system's mode shapes [17]. Therefore, the initial degrees of freedom can be mapped by the mode shape vectors of the system to a new set of degrees of freedom as follows:

$$\begin{Bmatrix} x_l \\ x_m \end{Bmatrix} = \{\phi\}_1 z_1(t) + \{\phi\}_2 z_2(t) \quad (5)$$

$$\{\phi\}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \{\phi\}_2 = \begin{Bmatrix} 1 \\ -\mu \end{Bmatrix} \quad (6)$$

Which here, $\{\phi\}_1$ and $\{\phi\}_2$ are the mode shapes of the system. Because of the orthogonality characteristic of the mode shapes respect to mass and stiffness matrices [17], the original differential equation (eq. 1) can be split to the following independent equations.

$$(m_l + m_m) \ddot{z}_1(t) = F_m(t) \quad (7)$$

$$m_e \ddot{z}_2(t) + B_e \dot{z}_2(t) + k_e z_2(t) = -\mu F_m(t) \quad (8)$$

where the equivalent parameters used here, are defined as follows:

$$m_e = m_l(1 + \mu) \quad (9)$$

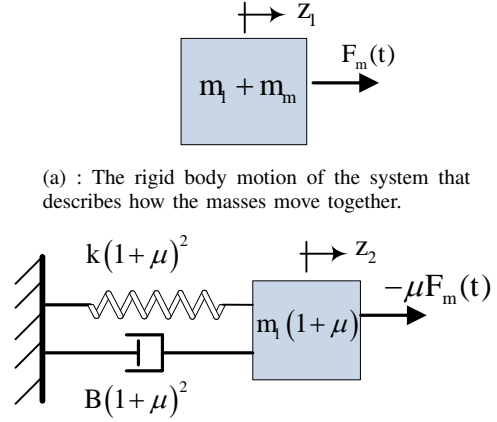
$$B_e = B(1 + \mu)^2 \quad (10)$$

$$k_e = k(1 + \mu)^2 \quad (11)$$

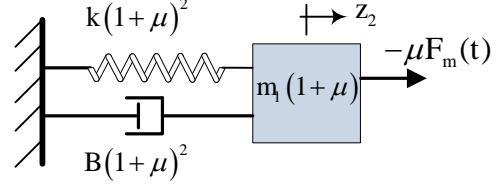
$$\mu = \frac{m_l}{m_m}. \quad (12)$$

The two new models demonstrated in Fig. 3 are the two new independent degrees of freedom. Fig. 3(a) represents the rigid body motion of the system (z_1) and describes how the masses move together. On the other hand, Fig. 3(b) describes the oscillation of the masses relative to each other (z_2). The whole response of the system is composed of a linear combination of these two independent motions as described by eq. 5.

Since we are looking for the case that gives us the largest final velocity, the motor should apply its maximum force from the beginning. It should be noted that here, like the case of jumping, the system of spring-damper can not sustain tension. Equation 7 can be easily solved by integrating that equation two times respect to time (t). On the other side, equation 8 is the well-known SDOF oscillation system [17]. The closed form solutions of the above equations are as follows:



(a) : The rigid body motion of the system that describes how the masses move together.



(b) : Equivalent oscillation of the masses relative to each other.

Fig. 3. The original system in Fig. 2 can be broken into two separate single degree of freedom systems. Fig. 3(a) and 3(a) illustrate a physical representation of the new systems.

$$z_1(t) = \frac{F_{max}}{2(m_l + m_m)} (t^2) \quad (13)$$

$$z_2(t) = -\frac{\mu F_{max}}{(1 + \mu)k_e} (1 - A(t)) \quad (14)$$

$$A(t) = e^{-\zeta w_e t} \left(\frac{\zeta}{\sqrt{1 - \zeta^2}} \sin(w_d t) + \cos(w_d t) \right) \quad (15)$$

Other parameters used in the above equations are:

$$w_e = \frac{k(1 + \mu)}{m_l} \quad (16)$$

$$\zeta = \frac{B_e}{2m_e w_e} \quad (17)$$

$$w_d = w_e \sqrt{1 - \zeta^2} \quad (18)$$

Here, w_e , ζ and w_d are respectively natural frequency, damping ratio and damped frequency of the equivalent system. The relative movement of the masses respect to each other (z_2 and \dot{z}_2) determines the contact between the object and the system. As the eq. 14 is like a constant force applying on a single degree of freedom system, the reaction force of the support never passes the zero line in the existence of damping. When there is no damping in the system, the dynamical force touches the zero line, but still never crosses that line. Fig. 4 shows how dynamical force of the system ($B \cdot \dot{z}_2 + k \cdot z_2$) varies respect to time. It can be concluded that as long as the motor applies the maximum force to the system, the object will not leave the system and consequently it means that the object is accelerated until some hard stops happen.

The first hard stop that should be controlled is the spring length limit. To control the adequacy of the stiffness and/or damping of the system to satisfy the spring limit length equation 14 is used. The required spring deflection is determined

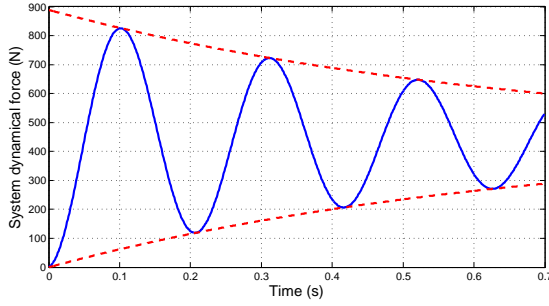


Fig. 4. Variation of the dynamical force of the spring-damper system to the object respect to time.

by the maximum possible value for the oscillation degree of freedom (z_2) which is represented in equation 8 (Fig. 3(b)). Therefore, the relation between the physical characteristics of the system and the maximum possible spring deflection can be obtained as follows:

$$l_{spring} = \frac{\mu F_{max}}{(1 + \mu)^2 k} (1 + e^{-\zeta\pi}) \quad (19)$$

It is the first and the simplest equation to calculate the required stiffness and/or damping of the system. The largest value for the required stiffness is when the damping is equal to zero. Also it can be understood that damping has not significant effect on limiting the spring compression.

The relative movement of the two masses shows if the object leaves the system or not. If the dynamic force applied by the spring and damper becomes zero, it means the object is on the onset of the separation. Based on the eq. 14, the relative equation of the masses is like a single degree of freedom under a constant force. The variation of this function is shown in Fig. 4

To satisfy other physical limitations of the actuator, two scenarios may happen. In the first scenario, the motor reaches its maximum length before reaching its maximum velocity and the second one is that motor reaches its maximum velocity before reaching its maximum length. For the first scenario, the system follows a single rule, but for the second one, the dynamics equation of the system alters when the motor reaches its maximum velocity (means the motor does not apply force when it is at its maximum velocity).

The velocities of the object and the motor can be obtained as follows:

$$\dot{x}_l(t) = \frac{F_{max}}{m_l + m_m} \left(t - \frac{1}{w_d} e^{-\zeta w_e t} \sin(w_d t) \right) \quad (20)$$

$$\dot{x}_m(t) = \frac{F_{max}}{m_l + m_m} \left(t + \frac{\mu}{w_d} e^{-\zeta w_e t} \sin(w_d t) \right) \quad (21)$$

The first term in the parenthesis above is common for both load velocity and motor velocity which shows the rigid body motion of the system. But, the second term shows the relative motion between the object and the motor. In the ideal situation where there was no motor limitations, we

just needed to find a stiffness correspond to maximizing the second term. In the real situation that we do have physical limitations like motor length limit and motor velocity limit, the first term (means the time that the motor drives the system) also influences the results and makes the analysis of the system far more complicated. To consider the motor length limit, we need the position of the motor at each instant. The positions of the load and the motor are:

$$x_l(t) = \frac{F_{max}}{m_l + m_m} \left(0.5t^2 - \frac{1}{w_e^2} (1 - A(t)) \right) \quad (22)$$

$$x_m(t) = \frac{F_{max}}{m_l + m_m} \left(0.5t^2 + \frac{\mu}{w_e^2} (1 - A(t)) \right) \quad (23)$$

$$A(t) = e^{-\zeta w_e t} \left(\frac{\zeta}{\sqrt{1 - \zeta^2}} \sin(w_d t) + \cos(w_d t) \right) \quad (24)$$

Since the motor can not travel more than its maximum length, the second equation above should be less than or equal to the maximum motor length. This equation relates the motor travel length limit to the physical characteristics of the system.

$$l_{motor} = \frac{F_{max}}{m_l + m_m} \left(0.5t_f^2 + \frac{\mu}{w_e^2} (1 - A(t_f)) \right) \quad (25)$$

Which $A(t_f)$ is given in eq. 24. This equation should be solved respect to the time (t_f) which shows the time of the process. After that, everything can be obtained by the equations 20 to 23.

V. SIMULATION

Because of the complicated form of the mathematical formulas, understanding the role of each parameter on the behavior of the system is not easy. In this section, based on the closed form solutions in the previous section, we present simulations to show the effect of the physical parameters on the behavior of the system by graphs. To accomplish the simulations, following characteristics are assumed for the system.

Parameter	Description	Value
m_m	Motor/transmission mass	5kg
m_l	Load mass	10kg
l_{spring}	Spring maximum length	1m
l_{motor}	Motor maximum length	1m
F_{max}	Motor maximum force	1000N
v_{max}	Motor maximum velocity	5 $\frac{m}{s}$

The two scenarios mentioned before are investigated here in simulation. In the first scenario, we assume that the motor can reach any velocity until the maximum length of the motor. For the second scenario, in addition to the motor length limit, the motor can not pass a predefined velocity.

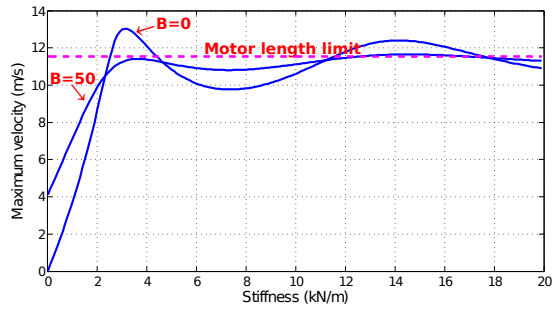


Fig. 5. The effect of spring stiffness on the maximum velocity of the object for the case of undamped ($B = 0$) and with damping equal to $B = 50$. The dashed straight line shows the velocity of the object when it is rigidly connected to the motor and no motor velocity limit is assumed.

A. Motor length determines the final velocity

In this scenario, we assume that the motor length limits the final velocity. Therefore, we can assume there is no limit on the motor velocity. It makes the equations simpler and gives us good information about the dynamics of the system. Figure 5 shows how damping and spring stiffness affect the maximum load velocity. The object gets its maximum velocity at the first peak. Also the largest value for the velocity is obtained when there is no damping in the system. The straight dashed line shows the velocity when the motor was directly connected to the object and it is accelerated until the motor hard stop occurs. Interesting note here is that, when for some reasons we could not provide the most optimum stiffness (like for example having a very small limit for spring length), there are some other choices far stiffer than that. If we have a tight spring limit concern we should not use any damper in the system, otherwise the behavior of the system will not be improved significantly.

To generate the response graphs in Fig. 5, the motor should at least provide the velocities shown in Fig. 6. In this figure, the maximum motor velocity for different spring stiffnesses is shown. As can be seen in both figures 5 and 6, to have the highest thrown velocity (which is about 13 m/s in this simulation) the motor should just have the maximum velocity about 8 m/s (means about 50 percent less). When the motor can not provide that velocity, the dynamic equation of the system changes and the behavior of the system will not be the same as was shown earlier. Interesting note in these figures is that, in about the same stiffness that the object gets its maximum velocity, the motor needs the smallest velocity.

B. Motor velocity limit determines the final velocity

The velocity limit on the motor alters the dynamics of the system. When the motor reaches its maximum velocity, it can no longer apply force to the system and the combined system travels with a constant velocity (but with relative motion respect to each other). This change in the dynamics of the system, changes the shape of the graphs in Fig. 5. The new response of the system is shown in Fig. 7. Two dashed straight lines in this graph show the velocities correspond to the motor length limit (top one) and motor velocity limit

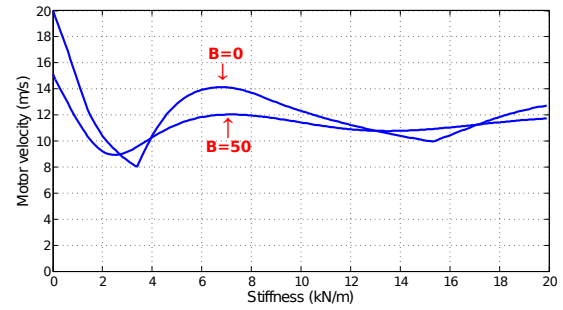


Fig. 6. The variations of maximum motor velocity respect to the stiffness of the spring in the case of undamped system ($B = 0$) and with damping equal to $B = 50$.

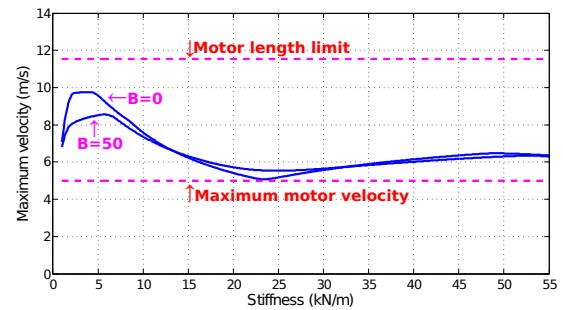


Fig. 7. Motor maximum velocity changes the shape of the response, but the optimum stiffness value remains optimum.

(bottom one). Adding elasticity to the system amplifies the maximum velocity of the object to about twice the value that it could have without the spring. Therefore, for the systems with low motor velocity limit, using appropriate spring and damper can improve the performance of the actuator significantly. The stiffness value corresponded to the highest velocity in Fig. 5 is still among the best choices for the stiffness. Moreover, the range of the spring stiffnesses that gives us the highest velocity was increased. Also it can be understood from the graph that damping has not significant effect on the response of the system for stiffnesses more than a certain value.

Another parameter that highly influences the final velocity of the object is the maximum force that can be provided by the motor. Figure 8 shows how the motor maximum force affects the final velocity of the thrown object. Based on the closed form solutions, the increase in velocity is linear.

For the case that we have velocity limit for the motor, the maximum motor force can not increase the final velocity beyond a certain value (Fig. 9). In Fig. 9 the responses of the system for three different motor forces are shown. When we have motor velocity limit, increasing the motor force can be not useful. Also, no matter what stiffness is chosen, the maximum achievable velocity will be constant. In these cases, if we need higher velocity, we have to increase the motor velocity limit.

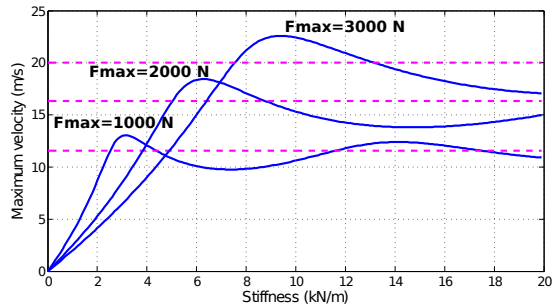


Fig. 8. The effect of the maximum motor force on the response of the system. Here, the system has no damping. As the maximum motor force increases the optimum required stiffness of the spring increases as well.

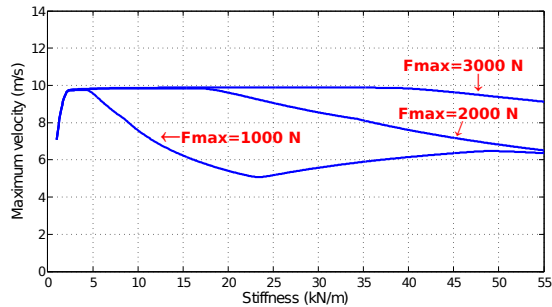


Fig. 9. The effect of the maximum motor force on the response of the system while we have motor velocity limit. The graphs are for non-damped cases. The highest velocity remains unchanged for different motor forces.

VI. CONCLUSIONS

In this paper, we extracted mathematical formulas for calculating the optimum stiffness and/or damping to be added to the passive dynamics of a system to enhance the performance of a throwing mechanism. By the use of the eigenvectors of the system, we simplified the equations to two independent single degree of freedom systems. Based on the extracted formulas and interpretations of the results from the simulation we can conclude that:

- 1) Adding elasticity and/or damping to the actuator if chosen accurately, improves the performance of the system. However, if these values are not used accurately, the outcome can even decrease the performance.
- 2) For each system, if we do not have motor velocity limit, there is a stiffness value that improves the performance of the system the most. Also, there are other stiffer values for the spring that improves the final velocity but not the same as the initial one.
- 3) The mechanical limitations like actuator range of motion directly affect the required stiffness of the spring.
- 4) Motor velocity is minimized for the stiffness that gives the object the largest velocity.
- 5) For the cases that we have low maximum motor velocity, adding spring-damper system can improve the performance of the system.
- 6) When the motor has low maximum motor velocity, increasing the motor force to improve the final velocity

of the object, is not helpful. In this case, it is recommended to use gearbox to increase the motor velocity instead of the motor force.

We defined relationships between series stiffness, series damping, drive system inertia, drive system torque limits, the drive velocity limit and the maximum velocity that the object can achieve using the mathematical model shown in Fig. 2. The linear spring was chosen because of its simplicity and its similarity to the SLIP (Spring-Load Inverted Pendulum) model used in legged locomotion. For future work, we plan to investigate the effect of the nonlinearity of the spring on the behavior of the system.

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